JOURNAL OF APPROXIMATION THEORY 15, 366–370 (1975)

L_p Convergence of Generalized Convex Functions and Their Uniform Convergence

O. SHISHA

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881

AND

S. TRAVIS

Code WA21, Naval Underwater Systems Center, Newport, Rhode Island 02840

1. Recently, Lewis and Shisha [2] have proved the following.

THEOREM 1. *Suppose $0 < p < \infty$, k is a positive integer, f is a real function continuous in (a, b) $(-\infty < a < b < \infty)$, and $(f_n)_{n=1}^{\infty}$ is a sequence of real functions, monotone increasing of order k on (a, b), namely,*

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_k \\ \vdots & \vdots & & \\ t_0^{k-1} & t_1^{k-1} & \cdots & t_k^{k-1} \\ f_n(t_0) & f_n(t_1) & \cdots & f_n(t_k) \end{vmatrix} \geqslant 0 \tag{1}$$

whenever $n \geqslant 1$ and $a < t_0 < t_1 \cdots < t_k < b$. Furthermore, suppose that $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p\, dx = 0$ (a Lebesgue integral). Then f_n converges uniformly to f in every $[c, d]$ with $a < c < d < b$.

When $k = 1$ ($k = 2$), Eq. (1) states that each f_n is increasing (convex) on (a, b). If $k \geqslant 2$, then [1, p. 381] (1) implies that for each n, $f_n^{(k-2)}$ exists and is continuous in (a, b).

Theorem 1 naturally raises the question: Can (1), which expresses convexity of the f_n with respect to $(t^j)_{j=0}^{k-1}$ [1, p. 375] be replaced by convexity of the f_n with respect to a general Tchebycheff system? The answer is positive and is the aim of the present article.

(2.) THEOREM 2. *Let $(u_j)_{j=0}^{k-1}$ $(k \geqslant 1)$ be a Tchebycheff system on (a, b)*

$(-\infty < a < b < \infty)$, *namely, each* u_j *is a real function, continuous in* (a, b), *and*

$$U\begin{pmatrix} u_0, u_1, \ldots, u_{k-1} \\ t_0, t_1, \ldots, t_{k-1} \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_{k-1}) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_{k-1}) \\ \vdots & & & \vdots \\ u_{k-1}(t_0) & u_{k-1}(t_1) & \cdots & u_{k-1}(t_{k-1}) \end{vmatrix} > 0 \qquad (2)$$

whenever $a < t_0 < t_1 \cdots < t_{k-1} < b$.

Suppose $0 < p < \infty$, f *is a real function, continuous in* (a, b), *and* $(f_n)_{n=1}^{\infty}$ *is a sequence of real functions, convex with respect to* $(u_j)_{j=0}^{k-1}$ *on* (a, b), *namely,*

$$U\begin{pmatrix} u_0, u_1, \ldots, u_{k-1}, f_n \\ t_0, t_1, \ldots, t_{k-1}, t_k \end{pmatrix} \geqslant 0 \qquad (3)$$

whenever $n \geqslant 1$ *and* $a < t_0 < t_1 \cdots < t_k < b$.

Suppose, again, $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p \, dx = 0$. *Then* f_n *converges uniformly to* f *in every* $[c, d]$ *with* $a < c < d < b$.

Proof. Assume the conclusion is false. Then there exist c, d $(a < c < d < b)$, $\epsilon > 0$, a subsequence of $(f_n)_{n=1}^{\infty}$, again denoted by $(f_n)_{n=1}^{\infty}$, and a sequence $(t_{k,n})_{n=1}^{\infty}$ of points in $[c, d]$ such that

$$|f_n(t_{k,n}) - f(t_{k,n})| \geqslant \epsilon \qquad \text{for all } n. \qquad (4)$$

Thus, $(t_{k,n})_{n=1}^{\infty}$ has a convergent subsequence (again denoted by $(t_{k,n})_{n=1}^{\infty}$) with a limit $t_k \epsilon [c, d]$. By (2), there must be a q, $0 \leqslant q \leqslant k-1$, with $u_q(t_k) \neq 0$. This, together with the continuity of f, assures the existence of δ, $a < t_k - \delta < t_k + \delta < b$ such that

$$|f(x) - f(y)| < \epsilon/(9k) \qquad \text{for all } x, y \text{ in } I = (t_k - \delta, t_k + \delta), \qquad (5)$$

and

$$u_q(y) \neq 0, \qquad |f(y)| \cdot |1 - \{u_q(x)/u_q(y)\}| < \epsilon/(9k) \qquad \text{for all } x, y \text{ in } I. \qquad (6)$$

Suppose $k > 1$. Set

$$t_j = t_k - (k-j)(k+1)^{-1}\,\delta, \qquad j = 0, \ldots, k-2$$

(so that, in particular, $t_k - \delta < t_0$) and choose $t_{k-1} \in (t_{k-2}, t_k)$ satisfying

$$U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_{k-1} \end{pmatrix} > \frac{1}{3} U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_k \end{pmatrix}; \tag{7}$$

$$U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_{k-1}, t_k \end{pmatrix} < U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_k \end{pmatrix} \tag{8}$$

$$(\text{one } t_j \text{ missing}) \quad 0 \leqslant j \leqslant k-2$$

(where (8) holds for all possible omissions). This choice is possible since

$$U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_k \end{pmatrix} > 0,$$

the left-hand side of (7), as a function of t_{k-1}, is continuous at $t_{k-1} = t_k$; and the left-hand side of (8), for each omission, $\to 0$ as $t_{k-1} \to t_k$.

Similarly, choose $t_{k+1} \in (t_k, t_k + \delta)$ such that

$$U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_{k+1} \end{pmatrix} > \frac{1}{3} U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_k \end{pmatrix}, \tag{9}$$

$$U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_k, t_{k+1} \end{pmatrix} < U\begin{pmatrix} u_0, \dots, u_{k-2}, u_{k-1} \\ t_0, \dots, t_{k-2}, t_k \end{pmatrix}, \tag{10}$$

$$(\text{one } t_j \text{ missing}) \quad 0 \leqslant j \leqslant k-2$$

where (10) holds for all possible omissions.

If $k = 1$, let t_0 be a point of $(t_1 - \delta, t_1)$ satisfying

$$(*) \qquad u_0(t_0) > \tfrac{1}{3} u_0(t_1),$$

and let t_2 be a point of $(t_1, t_1 + \delta)$ satisfying

$$(**) \qquad u_0(t_2) > \tfrac{1}{3} u_0(t_1).$$

Let $k \geqslant 1$. Choose $\delta_1 > 0$ so that

$$t_k - \delta < t_0 - \delta_1 < t_{k+1} + \delta_1 < t_k + \delta, \tag{11}$$

and the intervals $I_j = (t_j - \delta_1, t_j + \delta_1)$, $j = 0, \dots, k+1$, are mutually disjoint; (12)

if $r_j \in I_j$ $(j = 0, 1, \dots, k+1)$, then (7)–(10), with each t_j replaced by r_j, continue to hold in case $k > 1$ and $(*)$ and $(**)$ continue to hold in case $k = 1$. (13)

Observe, from (11), that $I_j \subset I$, for $j = 0, 1, \dots, k+1$.

There exists an $N_1 \geqslant 1$, such that if $n \geqslant N_1$ and $0 \leqslant j \leqslant k+1, j \neq k$, then there is a $t_{j,n} \in I_j$ with

$$| f_n(t_{j,n}) - f(t_{j,n})| < \epsilon/(9k). \tag{14}$$

Choose $n \geqslant N_1$ so that $t_{k,n} \in I_k$.

Case 1 (see (4)).

$$f_n(t_{k,n}) - f(t_{k,n}) \leqslant -\epsilon. \tag{15}$$

Consider

$$U\begin{pmatrix} u_0, u_1, \ldots, u_{k-1}, f_n \\ t_{0,n}, t_{1,n}, \ldots, t_{k-1,n}, t_{k,n} \end{pmatrix}$$

$$= \begin{vmatrix} u_0(t_{0,n}) & \cdots & u_0(t_{k-1,n}) & u_0(t_{k,n}) \\ u_1(t_{0,n}) & & u_1(t_{k-1,n}) & u_1(t_{k,n}) \\ \vdots & & \vdots & \vdots \\ u_{k-1}(t_{0,n}) & & u_{k-1}(t_{k-1,n}) & u_{k-1}(t_{k,n}) \\ f_n(t_{0,n}) - \dfrac{f(t_{k,n})}{u_q(t_{k,n})} u_q(t_{0,n}) & \cdots & f_n(t_{k-1,n}) - \dfrac{f(t_{k,n})}{u_q(t_{k,n})} u_q(t_{k-1,n}) & f_n(t_{k,n}) - f(t_{k,n}) \end{vmatrix} \tag{16}$$

If $0 \leqslant j \leqslant k+1, j \neq k$, then by (14), (5), and (6),

$$\left| f_n(t_{j,n}) - \frac{f(t_{k,n})}{u_q(t_{k,n})} u_q(t_{j,n}) \right|$$
$$\leqslant | f_n(t_{j,n}) - f(t_{j,n})| + | f(t_{j,n}) - f(t_{k,n})| + | f(t_{k,n})| \cdot \left| 1 - \frac{u_q(t_{j,n})}{u_q(t_{k,n})} \right|$$
$$< \epsilon/(3k). \tag{17}$$

Denote by u^* the minor of $f_n(t_{k,n}) - f(t_{k,n})$ in (16). By (2), and by (7) (or (*)), (8) and (13), the minor of every other entry of the last row of (16) is >0 and $<3u^*$. Thus, by (15) and (17),

$$U\begin{pmatrix} u_0, u_1, \ldots, u_{k-1}, f_n \\ t_{0,n}, t_{1,n}, \ldots, t_{k-1,n}, t_{k,n} \end{pmatrix} < -\epsilon u^* + k[\epsilon/(3k)]\, 3u^* = 0,$$

contradicting (3).

Case 2.

$$- f_n(t_{k,n}) + f(t_{k,n}) \leqslant -\epsilon. \tag{18}$$

We consider, this time,

$$U\begin{pmatrix} u_0, u_1, \dots, u_{k-2}, u_{k-1}, f_n \\ t_{0,n}, t_{1,n}, \dots, t_{k-2,n}, t_{k,n}, t_{k+1,n} \end{pmatrix} = U\begin{pmatrix} u_0, u_1, \dots, u_{k-2}, u_{k-1}, -f_n \\ t_{0,n}, t_{1,n}, \dots, t_{k-2,n}, t_{k+1,n}, t_{k,n} \end{pmatrix}$$

$$= \begin{vmatrix} u_0(t_{0,n}) & \cdots & u_0(t_{k-2,n}) & u_0(t_{k+1,n}) & u_0(t_{k,n}) \\ u_1(t_{0,n}) & \cdots & u_1(t_{k-2,n}) & u_1(t_{k+1,n}) & u_1(t_{k,n}) \\ \vdots & & \vdots & \vdots & \vdots \\ u_{k-1}(t_{0,n}) & \cdots & u_{k-1}(t_{k-2,n}) & u_{k-1}(t_{k+1,n}) & u_{k-1}(t_{k,n}) \\ -f_n(t_{0,n}) + \dfrac{f(t_{k,n})}{u_q(t_{k,n})} u_q(t_{0,n}) & \cdots & -f_n(t_{k-2,n}) + \dfrac{f(t_{k,n})}{u_q(t_{k,n})} u_q(t_{k-2,n}) & -f_n(t_{k+1,n}) + \dfrac{f(t_{k,n})}{u_q(t_{k,n})} u_q(t_{k+1,n}) & -f_n(t_{k,n}) + f(t_{k,n}) \end{vmatrix} \tag{19}$$

(with an obvious meaning if $k = 1$).

Denote by u^{**} the minor of $-f_n(t_{k,n}) + f(t_{k,n})$ in (19). By (9), (10) (or (**)), and (13), the absolute value of the minor of every other entry of the last row of (19) is $< 3u^{**}$. Thus, by (18) and (17),

$$U\begin{pmatrix} u_0, u_1, \dots, u_{k-2}, u_{k-1}, f_n \\ t_{0,n}, t_{1,n}, \dots, t_{k-2,n}, t_{k,n}, t_{k+1,n} \end{pmatrix} < -\epsilon u^{**} + k[\epsilon/(3k)]\, 3u^{**} = 0,$$

contradicting (3).

References

1. S. Karlin and W. J. Studden, "Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966.
2. J. T. Lewis and O. Shisha, *J. Approximation Theory* **14** (1975), 281–284.